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SOME CLASSES OF ORDER α FOR SECOND-ORDER DIFFERENTIAL INEQUALITIES

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ABSTRACT. For analytic functions $f(z)$ in the open unit disk \mathbb{U} with $f(0) = f'(0) - 1 = 0$, S. S. Miller and P. T. Mocanu (Integral Transform. Spec. Funct. **19**(2008)) have considered some sufficient problems for starlikeness. The object of the present paper is to discuss some sufficient problems for $f(z)$ to be in some classes of order α .

1. INTRODUCTION

Let \mathcal{A}_n denote the class of functions

$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots \quad (n = 1, 2, 3, \dots)$$

that are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A} = \mathcal{A}_1$. We denote by \mathcal{S} the subclass of \mathcal{A}_n consisting of univalent functions $f(z)$ in \mathbb{U} .

Let $\mathcal{S}^*(\alpha)$ be defined by

$$\mathcal{S}^*(\alpha) = \left\{ f(z) \in \mathcal{A}_n : \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \alpha, \ 0 \leq \exists \alpha < 1 \right\}.$$

We denote by $\mathcal{S}^* = \mathcal{S}^*(0)$. Also, let $\mathcal{C}(\alpha)$ be

$$\mathcal{C}(\alpha) = \{ f(z) \in \mathcal{A}_n : \operatorname{Re}(f'(z)) > \alpha, \ 0 \leq \exists \alpha < 1 \}.$$

We also denote by $\mathcal{C} = \mathcal{C}(0)$. Also, let $\mathcal{K}(\alpha)$ be defined by

$$\mathcal{K}(\alpha) = \left\{ f(z) \in \mathcal{A}_n : \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, \ 0 \leq \exists \alpha < 1 \right\}.$$

We denote by $\mathcal{K} = \mathcal{K}(0)$. From the definitions for $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, we know that $f(z) \in \mathcal{K}(\alpha)$ if and only if $zf'(z) \in \mathcal{S}^*(\alpha)$.

The basic tool in proving our results is the following lemma due to Jack [1] (also, due to Miller and Mocanu [3]).

Lemma 1. *Let the function $w(z)$ defined by*

$$w(z) = a_n z^n + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \dots \quad (n = 1, 2, 3, \dots)$$

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be analytic in \mathbb{U} with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_0 \in \mathbb{U}$, then there exists a real number $k \geq n$ such that

$$\frac{z_0 w'(z_0)}{w(z_0)} = k$$

and

$$\operatorname{Re} \left(\frac{z_0 w''(z_0)}{w'(z_0)} \right) + 1 \geq k.$$

2. MAIN RESULTS

Applying Lemma 1, we derive the following lemma.

Lemma 2. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\left| z f''(z) - \beta \left(f'(z) - \frac{f(z)}{z} \right) \right| < \rho |n + 1 - \beta| \quad (z \in \mathbb{U})$$

for some real $\rho > 0$ and some complex β with $\operatorname{Re}(\beta) < n + 1$, then

$$\left| f'(z) - \frac{f(z)}{z} \right| < \rho \quad (z \in \mathbb{U}).$$

Proof. Let us define $w(z)$ by

$$\begin{aligned} w(z) &= f'(z) - \frac{f(z)}{z} \\ &= n a_{n+1} z^n + (n+1) a_{n+2} z^{n+1} + \dots \quad (z \in \mathbb{U}). \end{aligned} \tag{1}$$

Then, clearly, $w(z)$ is analytic in \mathbb{U} and $w(0) = 0$. Differentiating both sides in (1), we obtain

$$z f''(z) = z w'(z) + w(z) \quad (z \in \mathbb{U}),$$

and therefore,

$$\begin{aligned} \left| z f''(z) - \beta \left(f'(z) - \frac{f(z)}{z} \right) \right| &= |z w'(z) + (1 - \beta) w(z)| \\ &= |w(z)| \left| \frac{z w'(z)}{w(z)} + 1 - \beta \right| \\ &< \rho |n + 1 - \beta| \quad (z \in \mathbb{U}). \end{aligned}$$

If there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = \rho,$$

then Lemma 1 gives us that $w(z_0) = \rho e^{i\theta}$ and $z_0 w'(z_0) = k w(z_0)$ ($k \geq n$). Thus we have

$$\begin{aligned} \left| z_0 f''(z_0) - \beta \left(f'(z_0) - \frac{f(z_0)}{z_0} \right) \right| &= |w(z_0)| \left| \frac{z_0 w'(z_0)}{w(z_0)} + 1 - \beta \right| \\ &= \rho |k + 1 - \beta| \\ &\geq \rho |n + 1 - \beta|. \end{aligned}$$

This contradicts our condition in the lemma. Therefore, there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = \rho$. This means that $|w(z)| < \rho$ for all $z \in \mathbb{U}$, that is, that

$$\left| f'(z) - \frac{f(z)}{z} \right| < \rho \quad (z \in \mathbb{U}).$$

□

Also applying Lemma 1, we have

Lemma 3. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\left| z f''(z) - \beta \left(f'(z) - \frac{f(z)}{z} \right) \right| < \rho n |n + 1 - \beta| \quad (z \in \mathbb{U})$$

for some real $\rho > 0$ and some complex β with $\operatorname{Re}(\beta) < n + 1$, then

$$\left| \frac{f(z)}{z} - 1 \right| < \rho \quad (z \in \mathbb{U}).$$

Proof. Let us define the function $w(z)$ by

$$\begin{aligned} w(z) &= \frac{f(z)}{z} - 1 \\ &= a_{n+1}z^n + a_{n+2}z^{n+1} + \dots \quad (z \in \mathbb{U}). \end{aligned}$$

Clearly, $w(z)$ is analytic in \mathbb{U} and $w(0) = 0$. We want to prove that $|w(z)| < \rho$ in \mathbb{U} . Since

$$z f''(z) = z^2 w''(z) + 2z w'(z) \quad (z \in \mathbb{U}),$$

we see that

$$\begin{aligned} \left| z f''(z) - \beta \left(f'(z) - \frac{f(z)}{z} \right) \right| &= |z^2 w''(z) + (2 - \beta)z w'(z)| \\ &< \rho n |n + 1 - \beta| \quad (z \in \mathbb{U}). \end{aligned}$$

If there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = \rho,$$

then Lemma 1 gives us that $w(z_0) = \rho e^{i\theta}$, $z_0 w'(z_0) = k w(z_0)$ ($k \geq n$) and

$$\operatorname{Re} \left(\frac{z_0 w''(z_0)}{w'(z_0)} \right) + 1 \geq k.$$

Thus we have

$$\begin{aligned}
\left| z_0 f''(z_0) - \beta \left(f'(z_0) - \frac{f(z_0)}{z_0} \right) \right| &= |z_0^2 w''(z_0) + (2 - \beta) z_0 w'(z_0)| \\
&= |z_0 w'(z_0)| \left| \frac{z_0 w''(z_0)}{w'(z_0)} + 2 - \beta \right| \\
&= \rho k \left| \frac{z_0 w''(z_0)}{w'(z_0)} + 2 - \beta \right| \\
&\geq \rho k \left| \operatorname{Re} \left(\frac{z_0 w''(z_0)}{w'(z_0)} \right) + 2 - \beta \right| \\
&\geq \rho k |k + 1 - \beta| \\
&\geq \rho n |n + 1 - \beta|.
\end{aligned}$$

This contradicts the condition in the lemma. Therefore, there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = \rho$. This means that $|w(z)| < \rho$ for all $z \in \mathbb{U}$. \square

From Lemma 2 and Lemma 3, we drive the following results for $\mathcal{S}^*(\alpha)$.

Theorem 1. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\left| z f''(z) - \beta \left(f'(z) - \frac{f(z)}{z} \right) \right| < \frac{(1 - \alpha)n|n + 1 - \beta|}{n + 1 - \alpha} \quad (z \in \mathbb{U})$$

for some real $0 \leq \alpha < 1$ and some complex β with $\operatorname{Re}(\beta) < n + 1$, then

$$\left| \frac{z f'(z)}{f(z)} - 1 \right| < 1 - \alpha \quad (z \in \mathbb{U}),$$

so that $f(z) \in \mathcal{S}^(\alpha)$.*

Proof. From Lemma 2 and Lemma 3, we have

$$\left| f'(z) - \frac{f(z)}{z} \right| < \frac{n(1 - \alpha)}{n + 1 - \alpha} \quad (z \in \mathbb{U}) \tag{2}$$

and

$$\left| \frac{f(z)}{z} - 1 \right| < \frac{1 - \alpha}{n + 1 - \alpha} \quad (z \in \mathbb{U}). \tag{3}$$

From (2) and (3),

$$\begin{aligned}
\frac{n(1 - \alpha)}{n + 1 - \alpha} &> \left| f'(z) - \frac{f(z)}{z} \right| \\
&= \left| \frac{f(z)}{z} \right| \left| \frac{z f'(z)}{f(z)} - 1 \right| \\
&> \left(1 - \frac{1 - \alpha}{n + 1 - \alpha} \right) \left| \frac{z f'(z)}{f(z)} - 1 \right| \\
&= \frac{n}{n + 1 - \alpha} \left| \frac{z f'(z)}{f(z)} - 1 \right| \quad (z \in \mathbb{U}).
\end{aligned}$$

So, we can get

$$\frac{n}{n+1-\alpha} \left| \frac{zf'(z)}{f(z)} - 1 \right| < \frac{n(1-\alpha)}{n+1-\alpha} \quad (z \in \mathbb{U}).$$

which completes the proof of the theorem. \square

When we put $f(z)$ by $zf'(z)$ in Theorem 1, we have

Corollary 1. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$|z^2 f'''(z) + (2-\beta)zf''(z)| < \frac{(1-\alpha)n|n+1-\beta|}{n+1-\alpha} \quad (z \in \mathbb{U})$$

for some real $0 \leq \alpha < 1$ and some complex β with $\operatorname{Re}(\beta) < n+1$, then

$$\left| \left(1 + \frac{zf''(z)}{f'(z)} \right) - 1 \right| < 1 - \alpha \quad (z \in \mathbb{U}),$$

so that $f(z) \in \mathcal{K}(\alpha)$.

Example 1. For some real $0 \leq \alpha < 1$ and some complex β with $\operatorname{Re}(\beta) < n+1$, we consider the function $f(z)$ given by

$$f(z) = z + \frac{1-\alpha}{n+1-\alpha} z^{n+1} \quad (z \in \mathbb{U}).$$

The function $f(z)$ satisfies Theorem 1.

Next, we consider $\mathcal{C}(\alpha)$.

Theorem 2. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$|zf''(z) - \beta(f'(z) - 1)| < (1-\alpha)|n-\beta| \quad (z \in \mathbb{U})$$

for some real $0 \leq \alpha < 1$ and some complex β with $\operatorname{Re}(\beta) < n$, then

$$|f'(z) - 1| < 1 - \alpha \quad (z \in \mathbb{U}).$$

This means that $f(z) \in \mathcal{C}(\alpha)$.

Proof. Define $w(z)$ in \mathbb{U} by

$$\begin{aligned} w(z) &= \frac{f'(z) - 1}{1 - \alpha} \\ &= \frac{(n+1)a_{n+1}}{1 - \alpha} z^n + \frac{(n+2)a_{n+2}}{1 - \alpha} z^{n+1} + \dots \quad (z \in \mathbb{U}). \end{aligned} \tag{4}$$

Evidently, $w(z)$ analytic in \mathbb{U} and $w(0) = 0$. We want to prove $|w(z)| < 1$. Differentiating (4) and simplifying, we obtain

$$zf''(z) = (1 - \alpha)zw'(z) \quad (z \in \mathbb{U}).$$

and, hence

$$\begin{aligned} |zf''(z) - \beta(f'(z) - 1)| &= |(1 - \alpha)zw'(z) - \beta(1 - \alpha)w(z)| \\ &= (1 - \alpha)|w(z)| \left| \frac{zw'(z)}{w(z)} - \beta \right| \\ &< (1 - \alpha)|n - \beta| \quad (z \in \mathbb{U}). \end{aligned}$$

If there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1,$$

then Lemma 1 gives us that $w(z_0) = e^{i\theta}$ and $z_0 w'(z_0) = kw(z_0)$ ($k \geq n$). Thus we have

$$\begin{aligned} |z_0 f''(z_0) - \beta(f'(z_0) - 1)| &= (1 - \alpha)|w(z_0)| \left| \frac{z_0 w'(z_0)}{w(z_0)} - \beta \right| \\ &= (1 - \alpha)|k - \beta| \\ &\geq (1 - \alpha)|n - \beta|. \end{aligned}$$

This contradicts our condition in the theorem. Therefore, there is no $z_0 \in \mathbb{U}$ such that $w(z_0) = 1$. This means that $|w(z)| < 1$ for all $z \in \mathbb{U}$. \square

Example 2. For some real $0 \leq \alpha < 1$ and some complex β with $\operatorname{Re}(\beta) < n$, we take

$$f(z) = z + \frac{1 - \alpha}{n + 1} z^{n+1} \quad (z \in \mathbb{U}).$$

Then, $f(z)$ satisfies Theorem 2.

We get the following lemma from Lemma 1.

Lemma 4. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$|zf''(z) - \beta(f'(z) - 1)| < \rho|n - \beta| \quad (z \in \mathbb{U})$$

for some real $\rho > 0$ and some complex β with $\operatorname{Re}(\beta) < n$, then

$$|f'(z) - 1| < \rho \quad (z \in \mathbb{U}).$$

Proof. Letting

$$\begin{aligned} w(z) &= f'(z) - 1 \\ &= (n + 1)a_{n+1}z^n + (n + 2)a_{n+2}z^{n+1} + \dots \quad (z \in \mathbb{U}), \end{aligned}$$

we see that $w(z)$ is analytic in \mathbb{U} and $w(0) = 0$. Noting that

$$zf''(z) = zw'(z) \quad (z \in \mathbb{U}),$$

we have

$$\begin{aligned} |zf''(z) - \beta(f'(z) - 1)| &= |zw'(z) - \beta w(z)| \\ &= |w(z)| \left| \frac{zw'(z)}{w(z)} - \beta \right| \\ &< \rho|n - \beta| \quad (z \in \mathbb{U}). \end{aligned}$$

If there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = \rho,$$

then Lemma 1 gives us that $w(z_0) = \rho e^{i\theta}$ and $z_0 w'(z_0) = kw(z_0)$ ($k \geq n$). Thus we have

$$\begin{aligned} |z_0 f''(z_0) - \beta(f'(z_0) - 1)| &= |w(z_0)| \left| \frac{z_0 w'(z_0)}{w(z_0)} - \beta \right| \\ &= \rho|k - \beta| \\ &\geq \rho|n - \beta| \end{aligned}$$

which contradicts our condition in the lemma. Therefore, there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = \rho$. This means that $|w(z)| < \rho$ for all $z \in \mathbb{U}$. \square

Using Lemma 4, we have next theorem.

Theorem 3. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$|zf''(z) - \beta(f'(z) - 1)| < \alpha|n - \beta| \quad (z \in \mathbb{U})$$

for some real $0 < \alpha \leq \frac{1}{2}$ and some complex β with $\operatorname{Re}(\beta) < n$, or

$$|zf''(z) - \beta(f'(z) - 1)| < (1 - \alpha)|n - \beta| \quad (z \in \mathbb{U})$$

for some real $\frac{1}{2} \leq \alpha < 1$ and some complex β with $\operatorname{Re}(\beta) < n$, then

$$\left| \frac{1}{f'(z)} - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha} \quad (z \in \mathbb{U}),$$

which implies that $f(z) \in \mathcal{C}(\alpha)$.

Proof. We can get

$$|f'(z) - 1| < \rho \quad (z \in \mathbb{U}). \quad (5)$$

for $0 < \alpha \leq \frac{1}{2}$ and $\rho = \alpha$, or $\frac{1}{2} \leq \alpha < 1$ and $\rho = 1 - \alpha$ from Lemma 4. Using (5), we have

$$|f'(z) - 2\alpha| < S < |f'(z)|$$

for $0 < \alpha \leq \frac{1}{2}$ and $S = 1 - \alpha$, or $\frac{1}{2} \leq \alpha < 1$ and $S = \alpha$. Thus we get

$$\begin{aligned} S \left| \frac{1}{f'(z)} - \frac{1}{2\alpha} \right| &< |f'(z)| \left| \frac{1}{f'(z)} - \frac{1}{2\alpha} \right| \\ &= \left| 1 - \frac{f'(z)}{2\alpha} \right| \\ &= \frac{1}{2\alpha} |f'(z) - 2\alpha| \\ &< \frac{S}{2\alpha} \quad (z \in \mathbb{U}). \end{aligned}$$

So we obtain

$$S \left| \frac{1}{f'(z)} - \frac{1}{2\alpha} \right| < \frac{S}{2\alpha} \quad (z \in \mathbb{U}).$$

□

Example 3. For some real $0 < \alpha \leq \frac{1}{2}$ and some complex β with $\operatorname{Re}(\beta) < n$, we consider the function $f(z)$ given by

$$f(z) = z + \frac{\alpha}{n+1} z^{n+1} \quad (z \in \mathbb{U}).$$

The function $f(z)$ satisfies Theorem 3.

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